On the ranks of the Conway groups $Co_2$ and $Co_3^*$

Faryad Ali † and Mohammed Ali Faya Ibrahim ‡
Department of Mathematics,
King Khalid University,
P.O. Box 9004, Abha,
Saudi Arabia.

Abstract

Let $G$ be a finite group and $X$ a conjugacy class of $G$. We denote $\text{rank}(G:X)$ to be the minimum number of elements of $X$ generating $G$. In the present article, we determine the ranks of the Conway groups $Co_2$ and $Co_3$. Computations were carried with the aid of computer algebra system GAP [18].

1 Introduction and Preliminaries

There has recently been some interest in generation of simple groups by their conjugate involutions. It is well known that sporadic simple groups are generated by three conjugate involutions (see [5]). If a group $G = \langle a, b \rangle$ is perfect and $a^2 = b^3 = 1$ then clearly $G$ is generated by three conjugate involutions $a$, $a^b$ and $a^{b^2}$ (see [6]). Moori [15] proved that the Fischer group $Fi_{22}$ can be generated by three conjugate involutions. The work of Liebeck and Shalev [12] shows that all but finitely many classical groups can be generated by three involutions. However, the problem of finding simple classical groups which can be generated by three conjugate involutions is still very much open. The generation of a simple group by its conjugate elements in this context is of some interest. Therefore, we concentrate on the generation of simple groups by their conjugate elements and investigate two sporadic simple groups.

Suppose that $G$ is a finite group and $X \subseteq G$. We denote the rank of $X$ in $G$ by $\text{rank}(G:X)$, the minimum number of elements of $X$ generating $G$. This paper focuses

* Dedicated to Professor Jamshid Moori on the occasion of his 60th birthday
† E-Mail: Fali@kku.edu.sa
‡ E-Mail: mafibraheem@kku.edu.sa
1 INTRODUCTION AND PRELIMINARIES

on the determination of rank($G;X$) where $X$ is a conjugacy class of $G$ and $G$ is a sporadic simple group.

Moori in [13], [14] and [15] proved that rank($Fi_{22};2A$) $\in \{5,6\}$ and rank($Fi_{22};2B$) $=$ rank($Fi_{22};2C$) = 3 where 2A, 2B and 2C are the conjugacy classes of involutions of the smallest Fischer group $Fi_{22}$ as presented in the ATLAS [3]. The work of Hall and Soicher [9] show that rank($Fi_{22};2A$) = 6. Moor [16] determined the ranks of the Janko groups $J_1$, $J_2$ and $J_3$. More recently, in [1], the authors investigated the ranks of Higman-Sims group $HS$ and McLaughlin group $McL$. In the present article we continue our study on the ranks of sporadic simple groups and determine the ranks of the Conway’s sporadic simple groups $Co_2$ and $Co_3$.

For basic properties of $Co_2$ and $Co_3$, character tables of these groups and their maximal subgroups etc. we use ATLAS [3] and [18]. For detailed information about the computational techniques used in this paper the reader is encouraged to consult [1], [8], [15], and [16].

Next we discuss some background material and introduce the notation. We adopt the same notation as in the above mentioned papers. In particular, if $G$ is a finite group, $C_1,C_2,\cdots ,C_k$ are the conjugacy classes of its elements and $g_k$ is a fixed representative of $C_k$, then $\Delta_G(C_1,C_2,\cdots ,C_k)$ denotes the number of distinct tuples $(g_1,g_2,\cdots ,g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1g_2\cdots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1,C_2,\cdots ,C_k)$ is the structure constant of $G$ for the conjugacy classes $C_1,C_2,\cdots ,C_k$ and can be computed from the character table of $G$ (see [11], p.45) by the following formula

$$\Delta_G(C_1,C_2,\cdots ,C_k) = \frac{|C_1||C_2|\cdots |C_{k-1}|}{|G|} \sum_{i=1}^{m} \chi_i(g_1)\chi_i(g_2)\cdots \chi_i(g_{k-1})\chi_i(g_k)$$

where $\chi_1,\chi_2,\cdots ,\chi_m$ are the irreducible complex characters of $G$. Also, $\Delta^*_G(C_1,C_2,\cdots ,C_k)$ denotes the number of distinct tuples $(g_1,g_2,\cdots ,g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1g_2\cdots g_{k-1} = g_k$ and $G = \langle g_1,g_2,\cdots ,g_{k-1} \rangle$. If $\Delta^*_G(C_1,C_2,\cdots ,C_k) > 0$, then we say that $G$ is $(C_1,C_2,\cdots ,C_k)$-generated. If $H$ any subgroup of $G$ containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1,C_2,\cdots ,C_{k-1},C_k)$ denotes the number of distinct tuples $(g_1,g_2,\cdots ,g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1g_2\cdots g_{k-1} = g_k$ and $(g_1,g_2,\cdots ,g_{k-1}) \leq H$ where $\Sigma_H(C_1,C_2,\cdots ,C_k)$ is obtained by summing the structure constants $\Delta_H(c_1,c_2,\cdots ,c_k)$ of $H$ over all $H$-conjugacy classes $c_1,c_2,\cdots ,c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

The ATLAS serves as a valuable source of information and we use the Atlas notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order $n$ in $G$ is denoted by $nX$. For examples, $2A$ represents the first conjugacy class of involutions in a group $G$. We will use the maximal subgroups and the permutations characters of $Co_2$ and $Co_3$ on the conjugates (right cosets) of the maximal subgroups listed in the ATLAS [3] extensively.
The following results will be crucial in determining the ranks of finite groups.

**Lemma 1** (Moori [16]) Let $G$ be a finite simple group such that $G$ is $(lX,mY,nZ)$-generated. Then $G$ is $(lX,lX,\ldots,lX,(nZ)^m)$-generated.

**Corollary 2** Let $G$ be a finite simple group such that $G$ is $(lX,mY,nZ)$-generated, then $\text{rank}(G : lX) \leq m$.

**Proof:** Immediately follows from Lemma 1.  

**Lemma 3** (Conder et al. [4]) Let $G$ be a simple $(2X,mY,nZ)$-generated group. Then $G$ is $(mY,mY,(nZ)^2)$-generated.

## 2 Ranks of $Co_2$

The Conway group $Co_2$ is a sporadic simple group of order $2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ with 11 conjugacy classes of maximal subgroups. It has 60 conjugacy classes of its elements including three conjugacy classes of involutions, namely $2A, 2B$ and $2C$. The group $Co_2$ acts primitively on a set $\Omega$ of 2300 points. The point stabilizer of this action is isomorphic to $U_6(2):2$ and the orbits have length 1, 891 and 1408. The permutation character of $Co_2$ on the cosets of $U_6(2):2$ is given by $\chi_{U_6(2):2} = 1a + 275a + 2024a$ For basic properties of $Co_2$ and computational techniques, the reader is encouraged to consult [1], [2], [8] and [19].

We now compute the rank of each conjugacy class of $Co_2$.

It is well known that every sporadic simple group can be generated by three involutions (see [6]). In the following lemmas we prove that $Co_2$ can be generated by three involutions $a, b, c \in 2X$ where $X \in \{A, B\}$.

**Lemma 4** $Co_2$ is $(2B, 2B, 2B, 23A)$-generated.

**Proof:** Simple computation show that the structure constant $\Delta_{Co_2}(2B, 2B, 2B, 23A) = 12696$. If $z$ is a fixed element of order 23 in $Co_2$ then there are 12696 distinct triples $(\alpha, \beta, \gamma)$ such $\{\alpha, \beta, \gamma\} \subset 2A$ and $\alpha\beta\gamma = z$. We observe that the only maximal subgroup of $Co_2$ which has order divisible by 23 is $M_{23}$ and $z$ is contained in a unique conjugate of $M_{23}$. Hence, any proper $(2B, 2B, 2B, 23A)$-subgroup of $Co_2$ must be in $M_{23}$. Furthermore, the $2A$-class is the only class which fuses to $2B$-class of $Co_2$ in $M_{23}$. It then follows that $\Sigma_{M_{23}}(2B, 2B, 2B, 23A) = 3174$. Thus the total contribution from $M_{23}$ to the distinct triples $(\alpha, \beta, \gamma)$ with $\{\alpha, \beta, \gamma\} \subset 2A$ and $\alpha\beta\gamma = z$ is equal to 3174. Thus we have
\[
\Delta_{Co_2}(2B, 2B, 2B, 23A) \geq \Delta_{Co_2}(2B, 2B, 2B, 23A) - \Sigma_{M_{23}}(2B, 2B, 2B, 23A) = 12696 - 3174 > 0.
\]
Hence $Co_2$ is $(2B, 2B, 2B, 23A)$-generated. □

**Lemma 5** \( \text{rank}(Co_2 : 2X) = 3 \) where \( X \in \{B, C\} \).

**Proof:** Let \( X \in \{B, C\} \). Ganief and Moori have shown in [8] that $Co_2$ is $(2C, 3A, 23A)$-generated. Thus, from the previous lemma and the above stated result from [8] together with application of Corollary 2 imply that \( \text{rank}(Co_2 : 2X) \leq 3 \). But the case \( \text{rank}(Co_2 : 2X) = 2 \) is not possible since if there are \( x, y \in 2X \) such that \( Co_2 = \langle x, y \rangle \), then \( Co_2 \cong D_{2n} \) where \( n = o(xy) \). This concludes that \( \text{rank}(Co_2 : 2X) = 3 \) whenever \( X \in \{B, C\} \). □

**Lemma 6** The group $Co_2$ is not $(2A, 2A, 2A, tX)$-generated for any conjugacy class \( tX \) in $Co_2$.

**Proof:** The group $Co_2$ acts on a 275-dimensional irreducible complex module $V$. Let \( d_{nX} = \dim(V/C_V(nX)) \), the co-dimension of the fix space (in \( V \)) of a representative in \( nX \). Using the character table of $Co_2$ we list in Table I, the values of \( d_{nX} \), for the conjugacy classes \( nX \).

<table>
<thead>
<tr>
<th>( d_{nX} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{2A} )</td>
</tr>
<tr>
<td>112</td>
</tr>
</tbody>
</table>

Set \( T = \{2A, 2B, 2C, 4B, 4C, 4D, 4E, 4F, 4G\} \). If the group $Co_2$ is $(2A, 2A, 2A, tX)$-generated, then by Scott’s theorem (see [4] and [17]) we must have

\[
d_{2A} + d_{2A} + d_{2A} + d_{tX} \geq 2 \times 275.
\]

However, it is clear from Table I that \( 3 \times d_{2A} + d_{tX} < 550 \) for each \( tX \in T \) and therefore $Co_2$ is not $(2A, 2A, 2A, tX)$-generated, for each \( tX \in T \).

Next suppose that \( tX \notin T \) then computing the structure constants, we see that

\[
\Delta_{Co_2}(2A, 2A, 2A, tX) < |C_{Co_2}(tX)|
\]

for each \( tX \notin T \) except for \( tX = 12H \). Now an application of Lemma 3.3 in [23] shows that $Co_2$ is not $(2A, 2A, 2A, tX)$-generated for any \( tX \notin T \) and \( tX \neq 12H \).

Finally we consider the only remaining case $(2A, 2A, 2A, 12H)$. For this case we have \( \Delta_{Co_2}(2A, 2A, 2A, 12H) = 72 \) and \( |C_{Co_2}(z)| = 48, z \in 12H \). In order to show that $Co_2$ is not $(2A, 2A, 2A, 12H)$-generated we construct the group $Co_2$ by using its "standard generators" given in [21] and also in [20]. The group $Co_2$ has a
22-dimensional irreducible representation over \( GF(2) \). Using this representation we generate \( Co_2 = \langle a, b \rangle \), where \( a \) and \( b \) are \( 22 \times 22 \) matrices over \( GF(2) \) with orders 2 and 5 respectively such that \( ab \) has order 28. Using \GAP, we see that \( a \in 2A, b \in 5A \) and \( ab \in 2A \). We produce \( x = b^{-3}ab^3, y = ((babab)^2b^{11})^7 \) and \( z = axy \) such that \( x, y \in 2A \) and \( z \in 12H \). Let \( H = \langle a, x, y \rangle \) then \( H < Co_2 \) with \(|H| = 1152 \). We compute that \( \Sigma_H(2A, 2A, 2A, 12H) = 72 \) and consequently \( \Delta_{Co_2}^*(2A, 2A, 2A, 12H) = 0 \). Hence \( Co_2 \) is not \((2A, 2A, 2A, 12H)\)-generated. This completes the proof.

\[ \text{Lemma 7} \quad \text{rank}(Co_2 : 2A) = 4. \]

\textbf{Proof:} Direct computation using \GAP [18] and the results of [8] we see that \( Co_2 = \langle x, y \rangle \) such that \( x \in 2A, y \in 4G \) with \( xy \in 23A \). Therefore, \( Co_2 \) is \((2A, 4G, 23A)\)-generated. Now, it follows by Corollary 2 that \( \text{rank}(Co_2 : 2A) \leq 4 \). Since \( \text{rank}(Co_2 : 2A) > 3 \) by the above lemma, the result follows.

\[ \text{Lemma 8} \quad \text{The group } Co_2 \text{ is } (2C, tX, 23A)\text{-generated where } tX \in \{3A, 3B, 4C, 4E\}. \]

\textbf{Proof:} We observe that the only maximal subgroup of \( Co_2 \) which has order divisible by 23 is \( M_{23} \) and \( M_{23} \cap 2C = \emptyset \). Hence

\[
\begin{align*}
\Delta_{Co_2}^*(2C, 3A, 23A) &= \Delta_{Co_2}(2C, 3A, 23A) = 69 > 0, \\
\Delta_{Co_2}^*(2C, 3B, 23A) &= \Delta_{Co_2}(2C, 3B, 23A) = 69 > 0, \\
\Delta_{Co_2}^*(2C, 4C, 23A) &= \Delta_{Co_2}(2C, 4C, 23A) = 345 > 0, \\
\Delta_{Co_2}^*(2C, 4E, 23A) &= \Delta_{Co_2}(2C, 4E, 23A) = 3128 > 0.
\end{align*}
\]

Thus \( Co_2 \) is \((2C, tX, 23A)\)-generated for any \( tX \in \{3A, 3B, 4C, 4E\} \).

\[ \text{Corollary 9} \quad \text{Let } tX \in \{3A, 3B, 4C, 4E\}. \text{ Then } \text{rank}(Co_2 : tX) = 2. \]

\textbf{Proof:} From the previous lemma we know that \( Co_2 \) is \((2C, tX, 23A)\)-generated for any \( tX \in \{3A, 3B, 4C, 4E\} \). Now result follows applying Lemma 3.

\[ \text{Lemma 10} \quad \text{The group } Co_2 \text{ is } (4X, 4X, 10A)\text{-generated where } X \in \{A, B\}. \]

\textbf{Proof:} The structure constants \( \Delta_{Co_2}(4A, 4A, 10A) = 125 \). The only maximal subgroups of \( Co_2 \) which have non-empty intersection with the classes \( 4A \) and \( 10A \) are, up to isomorphism, \( K_1 \cong (2_+^{1+6} \times 2^4) . A_8 \) and \( K_2 \cong 3_+^{1+4} . 2_+^{1+4} . S_5 \). Direct computation on \GAP shows that \( \Sigma_{K_1}(4A, 4A, 10A) = 5 \) and \( \Sigma_{K_2}(4A, 4A, 10A) = 5 \). It then follows that \( \Delta_{Co_2}^*(4A, 4A, 10A) \geq 125 - 1(5) - 4(5) > 0 \). Hence \( Co_2 \) is \((4A, 4A, 10A)\)-generated.

Similarly, the structure constant \( \Delta_{Co_2}(4B, 4B, 23A) = 989 \), and the only maximal subgroup of \( Co_2 \) which has an order divisible by 23 is \( M_{23} \) but \( M_{23} \cap 4B = \emptyset \). This implies that

\[
\Delta_{Co_2}^*(4B, 4B, 23A) = \Delta_{Co_2}(4B, 4B, 23A) = 989 > 0
\]

and \( Co_2 \) is \((4B, 4B, 23A)\)-generated.
Corollary 11 Let \( tX \in \{4A, 4B\} \). Then \( \text{rank}(Co_2 : tX) = 2 \).

Proof: This is clear from the previous lemma. \( \square \)

Lemma 12 If \( tX \in \{4D, 4F, 6C, 6D\} \) then \( \text{rank}(Co_2 : tX) = 2 \).

Proof: Again, the only maximal subgroup of \( Co_2 \) which has an element of order 23 is \( M_{23} \) but \( M_{23} \cap tX = \emptyset \) for every \( tX \in \{4D, 4F, 6C, 6D\} \). We obtained

\[
\Delta^*_C(2B, 4D, 23A) = \Delta^*_C(2B, 4D, 23A) = 23 > 0,
\]
\[
\Delta^*_C(2B, 4F, 23A) = \Delta^*_C(2B, 4F, 23A) = 92 > 0,
\]
\[
\Delta^*_C(2B, 6C, 23A) = \Delta^*_C(2B, 6C, 23A) = 92 > 0,
\]
\[
\Delta^*_C(2B, 6D, 23A) = \Delta^*_C(2B, 6D, 23A) = 115 > 0.
\]

Hence \( Co_2 \) is \((2B, tX, 23A)\)-generated where \( tX \in \{4D, 4F, 6C, 6D\} \) and we get that \( \text{rank}(Co_2 : tX) = 2 \) where \( tX \in \{4D, 4F, 6C, 6D\} \). \( \square \)

Theorem 13 If \( nX \notin \{1A, 2A, 2B, 2C\} \), then \( \text{rank}(Co_2 : nX) = 2 \).

Proof: Set \( K = \{3A, 3B, 4X\} \) where \( X \in \{A, B, C, D, E, F\} \). If \( nX \in K \) then \( \text{rank}(Co_2 : nX) = 2 \) by Lemmas 5 to 12.

Again direct computations using \texttt{GAP} and from the results of Ganief and Moori [8] we get that \( Co_2 \cong \langle a, b \rangle \) where \( a \in 2A \) and \( b \in \{5A, 5B, 6A, 6B, 6E, 7A, 8A, 8B, 8C, 8D, 8F, 9A, 11A, 12A, 23A, 23B\} \). Now for group \( Co_2 \) we have the following power maps \((12A)^2 = 6B, (12B)^2 = 6A, (12D)^2 = 6E, (12E)^2 = 6B, (12F)^2 = 6E, (12G)^2 = 6A, (12H)^2 = 6E, (14A)^2 = 7A, (14B)^2 = 7A, (14C)^2 = 7A, (15A)^3 = 5B, (15B)^3 = 5A, (15C)^3 = 5A, (16A)^2 = 8D, (16B)^2 = 8C, (18A)^2 = 9A, (20A)^2 = 10A, (20B)^2 = 10C, (24A)^2 = 12C, (24B)^2 = 12B, (28A)^4 = 7A, (30A)^2 = 15A, (30B)^2 = 15B \) and \((30C)^2 = 15C\). Using the above power maps together with information from [8], we obtain that \( Co_2 \) is \((2A, nX, mY)\)-generated for \( nX \notin \{1A, 2A, 2B, 2C\} \) with appropriate \( mZ \). Now applying Lemma 3, \( Co_2 \) is \((nX, nX, (mZ)^2)\)-generated for \( nX \notin \{1A, 2A, 2B, 2C\} \). Hence \( \text{rank}(Co_2 : nX) = 2 \) where \( nX \notin \{1A, 2A, 2B, 2C\} \).

3 Ranks of \( Co_3 \)

The smallest Conway group \( Co_3 \) is a sporadic simple group of order \( 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \) with 14 conjugacy classes of maximal subgroups. The group \( Co_3 \) has 42 conjugacy classes of its elements. It has two conjugacy classes of involutions, namely \( 2A \) and \( 2B \). For basic properties of \( Co_3 \) we refer readers to [2], [3] and [7].

Lemma 14 \( \text{rank}(Co_3 : 2X) = 3 \) where \( X \in \{A, B\} \).
Proof: Wolder [22] and Ganief and Moori [8] proved that $Co_3$ is a Hurwitz group by showing that $Co_3$ is $(2B, 3C, 7A)$-generated. So again by Corollary 2, we have $\text{rank}(Co_3 : 2B) \leq 3$. But $\text{rank}(Co_3 : 2B) = 2$ is not possible, because if $(x, y) = Co_3$ for some $x, y \in 2B$ then $Co_3 \cong D_{2n}$ with $o(xy) = n$. Thus $\text{rank}(Co_3 : 2B) = 3$.

Now for the rank of involution $2A$ in $Co_3$ we consider the triple $(2A, 3C, 23A)$. Using the character table of $Co_3$ we compute that $\Delta_{Co_3}(2A, 3C, 23A) = 46$. The only maximal subgroup of $Co_3$ containing elements of order $23$, up to isomorphism, is $M_{23}$. But the conjugacy class $3C$ has empty intersection with $M_{23}$. Thus $Co_3$ contains no proper $(2A, 3C, 23A)$-subgroup and we get $\Delta_{Co_3}^*(2A, 3C, 23A) = \Delta_{Co_3}(2A, 3C, 23A) > 0$. Now the result follows again applying Corollary 2.

Next we deal with the non-invocation conjugacy classes of $Co_3$.

Lemma 15 The group $Co_3$ is $(3A, 3A, 15A)$-generated.

Proof: The only maximal subgroups of $Co_3$ having non-empty intersection with the classes $3A$ and $15A$, up to isomorphisms, are $M_1 \cong M_{Cl, 2}$, $M_2 \cong 2.S_6(2)$, $M_3 \cong U_3(5):S_3$ and $M_4 \cong 3^{1+4}:4S_6$. By considering the permutation character values of $Co_3$ on these maximal subgroups and their fusion maps into $Co_3$ we obtain that $\Sigma_{M_1}(3A, 3A, 15A) = 6 = \Sigma_{M_2}(3A, 3A, 15A)$ and $\Sigma_{M_3}(3A, 3A, 15A) = 0 = \Sigma_{M_4}(3A, 3A, 15A)$. Hence

$$
\Delta_{Co_3}^*(3A, 3A, 15A) \geq \Delta_{Co_3}(3A, 3A, 15A) - \Sigma_{M_1}(3A, 3A, 15A) - \Sigma_{M_3}(3A, 3A, 15A) - \Sigma_{M_4}(3A, 3A, 15A)
$$

$$
= 46 - 1(6) - 1(6) > 0.
$$

This concludes that $Co_3$ is $(3A, 3A, 15A)$-generated.

Corollary 16 $\text{rank}(Co_3 : 3A) = 2$.

Proof: From the previous lemma we know that $Co_3$ is $(3A, 3A, 15A)$-generated and so we get that $\text{rank}(Co_3 : 3A) = 2$.

Lemma 17 The group $Co_3$ is $(4A, 4A, 23A)$-generated.

Proof: The only maximal subgroup of $Co_3$ which has an order divisible by $23$ is $M_{23}$ and $M_{23} \cap 4A = \emptyset$. Hence $\Delta_{Co_3}^*(4A, 4A, 23A) = \Delta_{Co_3}(4A, 4A, 23A) = 414 > 0$. This shows that $Co_3$ is $(4A, 4A, 23A)$-generated.

Lemma 18 The group $Co_3$ is $(4B, 4B, 23A)$-generated.

Proof: The only maximal subgroups of $Co_3$ which has an order divisible by $23$ is $M_{23}$. The $4a$ and $23a$ are the only classes of $M_{23}$ which fuse to $4B$ and $23A$ classes of $Co_3$ respectively. This implies that

$$
\Delta_{Co_3}^*(4B, 4B, 23A) \geq \Delta_{Co_3}(4B, 4B, 23A) - \Sigma_{M_{23}}(4A, 4B, 23A)
$$

$$
= 174846 - 7866 > 0,
$$

proving that $(4B, 4B, 23A)$ is a generating triple for $Co_3$. 

\qed
Lemma 19  \[ \text{rank}(Co_3 : 3B) = 2. \]

Proof: The group \( Co_3 \) is \((3B, 3B, 23A)\)-generated (see [8], Corollary 3.2). Hence \( \text{rank}(Co_3 : 3B) = 2. \)

Theorem 20  If \( nX \notin \{1A, 2A, 2B\} \) then \( \text{rank}(Co_3 : nX) = 2. \)

Proof: If \( nX \in \{3A, 3B, 4A, 4B\} \) then \( \text{rank}(Co_3 : nX) = 2 \) by the above Lemmas 14 to 19.

Direct computation using GAP and results from Ganief and Moori [8] show that \( Co_3 = \langle a, b \rangle \) where \( a \in 2A, \) and \( b \in nX \) with \( nX \in \{3C, 6A, 6B, 6C, 6D, 8A, 8B, 8C, 9A, 9B\} . \) Since the power maps of \( Co_3 \) yields \( (6E)^2 = 3C, (10A)^2 = 5A, (10B)^2 = 5B, (12A)^2 = 6A = (12B)^2, (12C)^2 = 6C, (14A)^2 = 7A, (15A)^3 = 5A, (15B)^3 = 5B, (18A)^3 = 6B, (20A)^4 = 5A = (20B)^4, (21A)^3 = 7A, (22A)^2 = 11B, (22B)^2 = 11A, (24A)^4 = 6A = (24B)^4 \) and \( (30A)^6 = 5A, \) we have \( Co_3 = \langle a, c \rangle \) where \( c \in \{6E, 10A, 10B, 12A, 12B, 12C, 14A, 15A, 15B, 18A, 20A, 20B, 21A, 22A, 22B, 24A, 24B, 30A\} . \)

Therefore \( Co_3 \) is \((2A, nX, mY)\)-generated where \( nX \in \{3C, 6A, 6B, 6C, 6D, 6E, 8A, 8B, 8C, 9A, 9B, 10A, 10B, 12A, 12B, 12C, 14A, 15A, 15B, 18A, 20A, 20B, 21A, 22A, 22B, 24A, 24B, 30A\} \) with appropriate \( mY. \) Hence \( \text{rank}(Co_3 : nX) = 2 \) where \( nX \notin \{1A, 2A, 2B\} . \)

Acknowledgements

The authors are grateful to the referee for his/her valuable and constructive remarks, particularly concerning proof of Lemma 6.

References


