Abstract: A group $G$ is said to be $(2, 3, t)$-generated if it can be generated by two elements $x$ and $y$ such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = t$. In this paper, we determine $(2, 3, t)$-generations of the Tits simple group $T \cong 2F_4(2)'$ where $t$ is divisor of $|T|$. Most of the computations were carried out with the aid of computer algebra system GAP [17].

Key–Words: Tits group $2F_4(2)'$, simple group, $(2, 3, t)$-generation, generator.

1 Introduction

A group $G$ is called $(2, 3, t)$-generated if it can be generated by an involution $x$ and an element $y$ of order 3 such that $o(xy) = t$. The $(2, 3)$-generation problem has attracted a wide attention of group theorists. One reason is that $(2, 3)$-generated groups are homomorphic images of the modular group $PSL(2, Z)$, which is the free product of two cyclic groups of order two and three. The motivation of $(2, 3)$-generation of simple groups also came from the calculation of the genus of finite simple groups [22].

Moore in [15] determined the $(2, 3, p)$-generations of the smallest Fischer group $F_{22}$. In [11], Ganiief and Moore established $(2, 3, t)$-generations of the third Janko group $J_3$. In a series of papers [1], [2], [3], [4], [5], [12] and [13], the authors studied $(2, 3)$-generation and generation by conjugate elements of the sporadic simple groups $C_{01}, C_{02}, C_{03}, He, HN, Suz, Ru, HS, McL, Th$ and $Fi_{23}$. The present article is devoted to the study of $(2, 3, t)$-generations for the Tits simple group $T$, where $t$ is any divisor of $|T|$. For more information regarding the study of $(2, 3, t)$-generations, generation by conjugate elements as well as computational techniques used in this article, the reader is referred to [11], [2], [3], [4], [5], [11], [15], [16] and [22].

The Tits group $T \cong 2F_4(2)'$ is a simple group of order $17971200 = 2^{11}3^55^2$. The group $T$ is a subgroup of the Rudvalis sporadic simple group $Ru$ of index $8120$. The group $T$ also sits maximally inside the smallest Fischer group $F_{22}$ with index $3592512$. The maximal subgroups of the Tits simple group $T$ was first determined by Tchakerian [19]. Later but independently, Wilson [20] also determined the maximal subgroups of the simple group $T$, while studying the geometry of the simple groups of Tits and Rudvalis.

For basic properties of the Tits group $T$ and information on its subgroups the reader is referred to [20], [19]. The ATLAS of Finite Groups [9] is an important reference and we adopt its notation for subgroups, conjugacy classes, etc. Computations were carried out with the aid of GAP [17].

2 Preliminary Results

Throughout this paper our notation is standard and taken mainly from [1], [2], [3], [4], [5], [11] and [11]. In particular, for a finite group $G$ with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and $g_k$ a fixed representative of $C_k$, we denote $\Delta(G) = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1g_2\ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can easily be computed from the character table of $G$ (see [14], p.45) by the following formula $\Delta_G(C_1, C_2, \ldots, C_k) = \prod_{l=1}^{\infty} |\chi_l(g_1, g_2, \ldots, g_{k-1})|^{e_{l-2}}$ where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of $G$. Further, let $\Delta^* (G) = \Delta^*_G(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1g_2\ldots g_{k-1} = g_k$ such that $G = (g_1, g_2, \ldots, g_{k-1})$. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then we say that $G$ is $(C_1, C_2, \ldots, C_k)$-generated.
If $H$ is any subgroup of $G$ containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \ldots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1}) \in (C_1 \times C_2 \times \ldots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $(g_1, g_2, \ldots, g_{k-1}) \leq H$ where $\Sigma_H(C_1, C_2, \ldots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of $H$ over all $H$-conjugacy classes $c_1, c_2, \ldots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k - 1$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to ATLAS [9]. A general conjugacy class of elements of order $n$ in $G$ is denoted by $nX$. For example $2A$ represents the first conjugacy class of involutions in a group $G$.

The following results in certain situations are very effective at establishing non-generations.

**Theorem 1** (Scott’s Theorem, [8] and [18]) Let $x_1, x_2, \ldots, x_m$ be elements generating a group $G$ with $x_1 x_2 \cdots x_n = 1_G$ and $V$ be an irreducible module for $G$ of dimension $n \geq 2$. Let $C_V(x_i)$ denote the fixed point space of $(x_i)$ on $V$, and let $d_i$ be the codimension of $V/C_V(x_i)$. Then $d_1 + d_2 + \cdots + d_m \geq 2n$.

**Lemma 2** ([8]) Let $G$ be a finite centerless group and suppose $lX, mY, nZ$ are $G$-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^*(G) = 0$ and therefore $G$ is not $(lX, mY, nZ)$-generated.

### 3 (2, 3, t)-Generations of Tits group

The Tits group $T \cong \mathbb{F}_4(2)'$ has 8 conjugacy classes of its maximal subgroups as determined by Wilson [20] and listed in the ATLAS [9]. The group $T$ has 22 conjugacy classes of its elements including 2 involutions namely $2A$ and $2B$.

In this section we investigate $(2, 3, t)$-generations for the Tits group $T$ where $t$ is a divisor of $|T|$. It is a well known fact that if a group $G$ is $(2, 3, t)$-generated simple group, then $1/2 + 1/3 + 1/t < 1$ (see [7] for details). It follows that for the $(2, 3, t)$-generations of the Tits simple group $T$, we only need to consider $t \in \{8, 10, 12, 13, 16\}$.

**Lemma 3** The Tits simple group $T$ is not $(2A, 3A, tX)$-generated for any $tX \in \{8A, 8B, 8C, 8D, 10A\}$.

**Proof.** For the triples $(2A, 3A, 8A)$ and $(2A, 3A, 8B)$ non-generation follows immediately since the structure constants $\Delta_T(2A, 3A, 8A) = 0$ and $\Delta_T(2A, 3A, 8B) = 0$.

The group $T$ acts on 78-dimensional irreducible complex module $V$. We apply Scott’s theorem (cf. Theorem 1) to the module $V$ and compute that

\[
\begin{align*}
d_{2A} &= \dim(V/C_V(2A)) = 32, \\
d_{3A} &= \dim(C/C_V(3A)) = 54, \\
d_{8C} &= \dim(V/C_V(8C)) = 68, \\
d_{8D} &= \dim(V/C_V(8D)) = 68, \\
d_{10A} &= \dim(V/C_V(10A)) = 68
\end{align*}
\]

Now, if the group $T$ is $(2B, 3A, tX)$-generated, where $tX \in \{8C, 8D, 10A\}$, then by Scott’s theorem we must have

\[
d_{2A} + d_{3A} + d_{tX} \geq 2 \times 78 = 156.
\]

However, $d_{2A} + d_{3A} + d_{tX} = 154$, and non-generation of the group $T$ by these triples follows.

**Lemma 4** The Tits simple group $T$ is $(2B, 3A, 8Z)$-generated, where $Z \in \{A, B, C, D\}$ if and only if $Z = A$ or $B$.

**Proof.** Our main proof will consider the following three cases.

**Case (2B, 3A, 8Z), where $Z \in \{A, B\}$:** We compute $\Delta_T(2B, 3A, 8Z) = 128$. Amongst the maximal subgroup of $T$, the only maximal subgroups having non-empty intersection with any conjugacy class in the triple $(2B, 3A, tZ)$ is isomorphic to $H \cong 2^2.[2^8]:S_3$. However $\Sigma_H(2B, 3A, 8Z) = 0$, which means that $H$ is not $(2B, 3A, 8Z)$-generated. Thus $\Delta^*_T(2B, 3A, 8Z) = \Delta_T(2B, 3A, 8Z) = 128 > 0$, and the $(2B, 3A, 8Z)$-generation of $T$, for $Z \in \{A, B\}$, follows.

**Case (2B, 3A, 8C):** The only maximal subgroups of the group $T$ that may contain $(2B, 3A, 8C)$-generated subgroups, up to isomorphism, are $H_1 \cong L_3(2); (2) \text{(two non-conjugate copies)}$ and $H_2 \cong 2^2.[2^8]:S_3$. Further, a fixed element $z \in 8C$ is contained in two conjugate subgroup of each copy of $H_1$ and in a unique conjugate subgroup of $H_2$. A simple computation using GAP reveals that $\Delta_T(2B, 3A, 8C) = 112$, $\Sigma_{H_1}(2B, 3A, 8C) = \Sigma_{L_3(2)}(2B, 3A, 8C) = 20$ and $\Sigma_{H_2}(2B, 3A, 8C) = 32$. By considering the maximal subgroups of $H_1 \cong L_3(2)$ and $H_2$, we see that no maximal subgroup of $H_1$ and $H_2$ is $(2B, 3A, 8C)$-generated and hence no proper subgroup of $H_1$ and $H_2$ is $(2B, 3A, 8C)$-generated. Thus,

\[
\begin{align*}
\Delta^*_T(2B, 3A, 8C) &= \Delta_T(2B, 3A, 8C) \\
&= -4\Sigma_{H_1}(2B, 3A, 8S) - \Sigma_{H_2}(2B, 3A, 8C) \\
&= 112 - 4(32) - 32 = 0.
\end{align*}
\]
Therefore, the Tits simple group \( T \) is not \((2B,3A,8C)\)-generated.

**Case** \((2B,3A,8D)\): In this case, 
\[ \Delta_T(2B,3A,8D) = 112. \]
We prove that Tits simple group \( T \) is not \((2B,3A,8D)\)-generated by constructing the \((2B,3A,8D)\)-generated subgroup of the group \( H \) explicitly. We use the "standard generators" of the group \( T \) given by Wilson in [21]. The group \( T \) has a 26-dimensional irreducible representation over \( \mathbb{G} \mathbb{F}(2) \). Using this representation we generate the Tits group \( T = \langle a, b \rangle \), where \( a \) and \( b \) are \( 26 \times 26 \) matrices over \( \mathbb{G} \mathbb{F}(2) \) with orders 2 and 3 respectively such that \( ab \) has order 13. Using \( \text{GAP} \), we see that \( a \in 2A, b \in 3A \). We produce \( c = (abab)^6, p = ababab^2ab^2, d = (acp)^6, x = p^{32}dp^{-16} \) such that \( c, d, x \in 2B, p \in 10A \) and \( x 8D \). Let \( H = \langle x, b \rangle \) then \( H \subset T \) with \( H \cong L_3(3) \). Since no maximal subgroup of \( H \) is \((2B,3A,8D)\)-generated, that is no proper subgroup of \( H \) is \((2B,3A,8D)\)-generated and we have \( \Sigma_H(2B,3A,8D) = 28 \) and \( z \in 8D \) is contained in exactly two conjugate subgroups of each copy of \( H \), we obtain that \( \Delta_H^*(2B,3A,8D) = 0 \). Hence the Tits simple group \( T \) is not \((2B,3A,8D)\)-generated. This completes the lemma. \( \blacksquare \)

**Lemma 5** The Tits group \( T \) is \((2B,3A,10A)\)-generated.

**Proof.** Up to isomorphism, the only maximal subgroups having non-empty intersection with any conjugacy class in the triple \((2B,3A,10A)\) are isomorphic to \( H \cong 2^2 \cdot [2]^2 \cdot S_3, K \cong A_6 \cdot 2^2 \cdot (two \ non-conjugate \ copies) \). Since \( \Delta_T(2B,3A,10A) = 100 \) and \( \Sigma_H(2B,3A,10A) = 0 = \Sigma_K(2B,3A,10A) \), we conclude that no maximal subgroup of \( T \) is \((2B,3A,10A)\)-generated. Thus
\[ \Delta_T^*(2B,3A,10A) = \Delta_T(2B,3A,10A) = 100 \]
and the \((2B,3A,10A)\)-generation of Tits group \( T \) follows. \( \blacksquare \)

**Lemma 6** The Tits group \( T \) is not \((2X,3A,12Z)\)-generated where \( X, Z \in \{A, B\} \).

**Proof.** First we consider the case \( X = A \). The maximal subgroups of the group \( T \) that may contain \((2A,3A,12Z)\)-generated subgroups are isomorphic to \( H \cong 2^2 \cdot [2]^2 \cdot S_3 \) and \( K \cong 5^2 \cdot 4A_4 \) (two non-conjugate copies). We compute that \( \Delta_T(2A,3A,12Z) = 32, \Sigma_H(2A,3A,12Z) = 12 \) and \( \Sigma_K(2A,3A,12Z) = 15 \). A fixed element of order 12 in \( T \) is contained in a unique conjugate subgroup of \( H \) and two conjugate subgroups of \( K \). Since no maximal subgroup of each \( H \) and \( K \) is \((2A,3A,12Z)\)-generated, we obtain
\[ \Delta_T^*(2A,3A,12Z) = \Delta_T(2A,3A,12Z) \]
\[ -\Sigma_H^*(2A,3A,12Z) \]
\[ -4\Sigma_K^*(2A,3A,12Z) \]
\[ = 32 - 12 - 2(15) < 0 \]
and the non-generation of the group Tits by the triple \((2A,3A,12Z)\) follows.

Next, suppose That \( X = B \). There are six maximal subgroups of the group \( T \) having non-empty intersection with each conjugacy class in the triple \((2B,3A,12Z)\), are isomorphic to \( H = L_3(3) : 2 \) (two non-conjugate copies), \( H_2 \cong L_3(25), H_3 \cong 2^2 \cdot [2]^3 ; S_3 \) and \( H_4 = 5^2 : 4A_4 \) (two non-conjugate copies). Further, a fixed element of order 12 in \( T \) group is contained in a unique conjugate subgroup of each of \( H_1, H_2, H_3 \) and \( H_4 \). We calculate \( \Delta_T(2B,3A,12Z) = 84, \Sigma_{H_1}(2B,3A,12Z) = 27, \Sigma_{H_2}(2B,3A,12Z) = 24, \Sigma_{H_3}(2B,3A,12Z) = 12 \) and \( \Sigma_{H_4}(2B,3A,12Z) = 0 \). Since no maximal subgroup of each of the groups \( H_1, H_2, H_3 \) and \( H_4 \) is \((2B,3A,12Z)\)-generated. We conclude that
\[ \Delta_T^*(2B,3A,12Z) = \Delta_T(2B,3A,12Z) \]
\[ -2\Sigma_{H_1}^*(2B,3A,12Z) \]
\[ -\Sigma_{H_2}^*(2B,3A,12Z) \]
\[ -\Sigma_{H_3}^*(2B,3A,12Z) \]
\[ = 84 - 2(27) - 24 - 12 < 0 \]
Therefore Tits group \( T \) is not \((2B,3A,12Z)\)-generated. This completes the proof. \( \blacksquare \)

**Lemma 7** The Tits group \( T \) is \((2X,3A,13Z)\)-generated where \( X, Z \in \{A, B\} \) if and only if \( X = A \)

**Proof.** First we consider the case \( X = A \). The structure constant \( \Delta_T(2A,3A,13Z) = 13 \). The fusion maps of the maximal subgroup of Tits group \( T \) into the group \( T \) shows that there is no maximal subgroup of \( T \) has non-empty intersection with the classes in the triple \((2A,3A,13Z)\). That is no maximal subgroup of \( T \) is \((2A,3A,13Z)\)-generated. Hence,
\[ \Delta_T^*(2A,3A,13Z) = \Delta_T(2A,3A,13Z) = 13 > 0 \]
which implies that the Tits group \( T \) is \((2A,3A,13Z)\)-generated for \( Z \in \{A, B\} \).

Next suppose that \( X = B \). Up to isomorphism, the only maximal subgroups of \( T \) having non-empty intersection with each conjugacy class in the triple \((2B,3A,13Z)\) are isomorphic to \( L_3(3) : 2 \) (two non-conjugate copies) and \( L_2(25) \).
Further a fixed element of order 13 in the Tits group $T$ is contained in a unique conjugate of each of $L_3(3)2$ and in three conjugate of $L_2(25)$ subgroups. We compute that $\Delta_T(2B,3A,13Z) = 104$, $\Sigma_{L_3(3)2}(2B,3A,13Z) = \Sigma_{L_3(3)}(2B,3A,13A) = 13$ and $\Sigma_{L_2(25)}(2B,3A,13Z) = 26$. Now by considering the maximal subgroups of $L_3(3)$ and $L_2(25)$, we see that no maximal subgroup of the groups $L_3(3)$ and $L_2(25)$ is $(2B,2A,13Z)$-generated. It follows that no proper subgroup of $L_3(3)$ or $L_2(25)$ is $(2B,3A,13Z)$-generated. Thus we have

$$\Delta_T^*(2B,3A,13Z) = \Delta_T(2B,3A,13Z) - 2\Sigma_{L_3(3)}(2B,3A,13Z) - 3\Sigma_{L_2(25)}(2B,3A,13Z) = 104 - 2(13) - 3(26) - 12 = 0,$$

proving non-generation of the Tits group $T$ by the triple $(2B,3A,13Z)$, where $Z \in \{A,B\}$. □

Lemma 8 The Tits group $T$ is $(2X,3A,16Z)$-generated, where $X \in \{A,B\}$ and $Z \in \{A,B,C,D\}$.

Proof. We treat two cases separately.

Case $(2A,3A,16Z)$: The structure constant $\Delta_T(2A,3A,16Z) = 16$. We observe that the group isomorphic to $2^2(2^8);S_3$ is the only maximal subgroup of $T$ that may contain $(2A,3A,16Z)$-generated subgroups. However we calculate $\Sigma_H(2A,3A,16Z) = 0$ for $H \cong 2^2(2^8);S_3$ and hence $\Delta_T^*(2A,3A,16Z) = \Delta_T(2A,3A,16Z) = 16 > 0$, proving that $(2A,3A,16Z)$ is a generating triple of the Tits group.

Case $(2B,3A,16Z)$: Up to isomorphism, $H \cong 2^2(2^8);S_3$ is the only one maximal subgroup of $T$ that may admit $(2B,3A,16Z)$-generated subgroups. A fixed element of order 16 in the Tits group $T$ is contained in a unique conjugate subgroups of $H$. Since $\Delta_T(2B,3A,16Z) = 112$, $\Sigma_H(2B,3A,16Z) = 32$, we conclude that

$$\Delta_T^*(2B,3A,16Z) \geq 112 - 32 = 80 > 0$$

and the $(2B,3A,16Z)$-generation of $T$ follows. □

We now summarize our results in the next theorem.

Theorem 9 Let $tX$ be a conjugacy class of the Tits simple group $T$. The group $T$ is $(2A,3A,tX)$-generated if and only if $tX \in \{13Y,16Z\}$ where $Y \in \{A,B\}$ and $Z \in \{A,B,C,D\}$. Further, the group $T$ is $(2B,3A,tX)$-generated if and only if $tX \in \{8Y,10A,16Z\}$.

Proof. This is merely a restatement of the lemmas in this section. □

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